Journal of Applied Mathematics and Mechanics 73 (2009) 77–87

Contents lists available at ScienceDirect

Journal of Applied Mathematics and Mechanics

journal homepage: www.elsevier.com/locate/jappmathmech

The vibrations and stability of a prestressed plate on an elastic foundation☆

P.Ye. Tovstik

St Petersburg, Russia

ARTICLE INFO

Article history:
Received 27 November 2007

ABSTRACT

The free vibrations of a transversely isotropic prestressed linear elastic half-space, localized close to a free surface, are considered. The free vibrations of a prestressed transversely isotropic infinite plate, lying on an elastic foundation, are also considered. The dispersion equation is analysed as a function of the wave numbers, the elastic properties of the foundation and of the plate and the values of the prestresses. The investigation is confined to cases when the initial stresses are less than the critical values, while the elastic waves do not penetrate into the depth of the foundation but are localized close to the free surface. The stability of the half-space and the plate on an elastic foundation is also considered. When analysing the vibrations and the stability of the plate, the results in the three-dimensional formulation of the problem are compared with the results of the two-dimensional Kirchhoff–Love and Timoshenko–Reissner models.

1. The equations of motion of the foundation

The equations of motion of a prestressed orthotropic half-space have the form

$$\frac{\partial\sigma_{ij}}{\partial x_j}+\Delta_\sigma u_i-\rho\frac{\partial u_i}{\partial t^2} = 0, \quad \Delta_\sigma u_i = \sigma_j^0\frac{\partial^2 u_i}{\partial x_j^2}, \quad i, j = 1, 2, 3 \tag{1.1}$$

where $-\infty < x_1, x_2 < \infty$, $-\infty \le x_3 = z \le 0$ are Cartesian coordinates, u_1, u_2, $u_3 = w$ are the projections of the displacement, σ_j^0 are the initial stresses (negative for compression and assumed constant), σ_{ij} are additional stresses, ρ is the density of the material and t is the time.

For an orthotropic material the elasticity relations

$$\sigma_{ii} = E_{i1}\varepsilon_{11}+E_{i2}\varepsilon_{22}+E_{i3}\varepsilon_{33}, \quad \sigma_{ij} = G_{ij}\varepsilon_{ij}, \quad i\ne j$$
$$\varepsilon_{ii} = \frac{\partial u_i}{\partial x_i}, \quad \varepsilon_{ij} = \frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i} \tag{1.2}$$

by virtue of the equalities $E_{ij} = E_{ji}$, $G_{ij} = G_{ji}$ contain nine elastic constants.

For the transversely isotropic material considered below, the number of constants is reduced to five (E, E', G_{13}, ν, ν'), where[1]

$$E_{11} = E_{22} = \frac{E(1-\hat{\nu}^2)}{(1+\nu)c}, \quad E_{12} = \frac{E(\nu+\hat{\nu}^2)}{(1+\nu)c}, \quad E_{13} = E_{23} = \frac{E\nu'}{c}, \quad E_{33} = \frac{E'(1-\nu)}{c}$$
$$G_{12} = G = \frac{E}{2(1+\nu)}, \quad G_{13} = G_{23}, \quad \hat{\nu}^2E' = (\nu')^2E, \quad c = 1-\nu-2\hat{\nu}^2 \tag{1.3}$$

☆ *Prikl. Mat. Mekh.* Vol. 73, No. 1, pp. 106–120, 2009.

E-mail address: peter.tovstik@vail.ru.

0021-8928/$ – see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.jappmathmech.2009.03.005

We have the following relations

$$E_{11} = E_{12} + 2G_{12}, \quad E_{11}^0 = E_{11} - \frac{E_{13}^2}{E_{33}} = \frac{E}{1-\nu^2}, \quad E_{12}^0 = E_{12} - \frac{E_{13}^2}{E_{33}} = \frac{E\nu}{1-\nu^2} \tag{1.4}$$

For an isotropic body, in formulae (1.3) $E' = E$, $G_{13} = G$, $\nu' = \hat{\nu} = \nu$, where E is Young's modulus and ν is Poisson's ratio.

When choosing the moduli E_{ij} and G_{ij} for a transversely isotropic body we will proceed as follows. We will consider a multilayer body with alternating soft and hard isotropic plane layers with parameters

$$E_n, \nu_n, h_n, \quad n = 1, 2 \tag{1.5}$$

Averaging of the elastic properties over the thickness of the layers when h_1, $h_2 \to 0$, h_1/h_2 leads to a transversely isotropic material. When calculating its moduli E_{ij} and G_{ij} we will assume that the strain $\varepsilon_{11}, \varepsilon_{12}, \varepsilon_{22}$ and stresses $\sigma_{13}, \sigma_{23}, \sigma_{33}$ are continuous on the boundary of the layers. As a result, we obtain

$$hZ_2E_{11} = h_1h_2(E_2^2(1+\nu_1)^2(1-2\nu_1) + E_1^2(1+\nu_2)^2(1-2\nu_2)) +$$
$$+ E_1E_2((h_1^2+h_2^2)(1-\nu_1^2)(1-\nu_2^2) + 2h_1h_2\nu_1\nu_2(1+\nu_1)(1+\nu_2))$$
$$hZ_2E_{12} = h_1h_2(E_2^2(1+\nu_1)^2(1-2\nu_1)\nu_2 + E_1^2(1+\nu_2)^2(1-2\nu_2)\nu_1) +$$
$$+ E_1E_2(1+\nu_1)(1+\nu_2)((h_1^2\nu_1(1-\nu_2) + h_2^2\nu_2(1-\nu_1) + 2h_1h_2\nu_1\nu_2)$$
$$Z_1E_{13} = E_1E_2(h_1\nu_1(1-\nu_2) + h_2\nu_2(1-\nu_1))$$
$$\frac{h}{E_{33}} = \frac{h_1(1+\nu_1)(1-2\nu_1)}{E_1(1-\nu_1)} + \frac{h_2(1+\nu_2)(1-2\nu_2)}{E_2(1-\nu_2)}$$
$$\frac{h}{G_{13}} = \frac{2h_1(1+\nu_1)}{E_1} + \frac{2h_2(1+\nu_2)}{E_2}, \quad h = h_1 + h_2$$
$$Z_k = h(E_2h_1(1+\nu_1)^k(1-2\nu_1)(1-\nu_2^k) + E_1h_2(1+\nu_2)^k(1-2\nu_2)(1-\nu_1^k)), \quad k = 1, 2 \tag{1.6}$$

Unlike the similar formulae for shells (see Ref. 2), here we do not assume that $\sigma_{33} = 0$, which is why formulae (1.6) are so complicated.

2. Construction of the solution, localized close to the free surface

The doubly periodic solution of system (1.1) with wave numbers r_1 and r_2, localized close to the free surface $z = 0$, will be sought in the form

$$u_1 = u_1(z)\cos(r_1x_1)\sin(r_2x_2)e^{i\omega t}, \quad \sigma_{13} = \sigma_{13}(z)\cos(r_1x_1)\sin(r_2x_2)e^{i\omega t}$$
$$u_2 = u_2(z)\sin(r_1x_1)\cos(r_2x_2)e^{i\omega t}, \quad \sigma_{23} = \sigma_{23}(z)\sin(r_1x_1)\cos(r_2x_2)e^{i\omega t}$$
$$u_3 = u_3(z)\sin(r_1x_1)\sin(r_2x_2)e^{i\omega t}, \quad \sigma_{12} = \sigma_{12}(z)\cos(r_1x_1)\cos(r_2x_2)e^{i\omega t}$$
$$\{\sigma_{11}, \sigma_{22}, \sigma_{33}\} = \{\sigma_{11}(z), \sigma_{22}(z), \sigma_{33}(z)\}\sin(r_1x_1)\sin(r_2x_2)e^{i\omega t} \tag{2.1}$$

where ω is the frequency of free vibrations. The attenuation with distance from the surface $z = 0$ has the form

$$\{u_i(z), \sigma_{ij}(z)\} \to 0 \quad \text{as} \quad z \to -\infty \tag{2.2}$$

Taking (2.1) into account, system (1.1) can be reduced to a system of ordinary differential equations

$$G_{k3}^* u_k'' - ((E_{kk} + \sigma_k^0)r_k^2 + (G_{12} + \sigma_{3-k}^0)r_{3-k}^2 - \lambda)u_k - (E_{12} + G_{12})r_1r_2u_{3-k} +$$
$$+ (G_{k3} + E_{k3})r_ku_3' = 0, \quad k = 1, 2$$
$$E_{33}^* u_3'' - ((G_{13} + \sigma_1^0)r_1^2 + (G_{23} + \sigma_2^0)r_2^2 - \lambda)u_3 + -(E_{13} + G_{13})r_1u_1' - (E_{23} + G_{23})r_2u_2' = 0 \tag{2.3}$$

where

$$\lambda = \rho\omega^2, \quad G_{k3}^* = G_{k3} + \sigma_3^0, \quad k = 1, 2, \quad E_{33}^* = E_{33} + \sigma_3^0, \quad (\ )' = d(\ )/dz \tag{2.4}$$

while the expressions for the stresses have the form

$$\sigma_{k3} = G_{k3}(u_k' + r_k u_3), \quad k = 1, 2, \quad \sigma_{33} = -E_{13}r_1u_1 - E_{23}r_2u_2 + E_{33}u_3' \tag{2.5}$$

For a transversely isotropic material, system (2.3), after making the replacements

$$u = (r_1u_1 + r_2u_2)/r, \quad \upsilon = (r_2u_1 - r_1u_2)/r,$$
$$\sigma = (r_1\sigma_{13} + r_2\sigma_{23})/r, \quad \tau = (r_2\sigma_{13} - r_1\sigma_{23})/r, \quad r^2 = r_1^2 + r_2^2 \tag{2.6}$$

can be decomposed into fourth-order and second-order systems

$$G_{13}^* u'' - (E_{11} + \sigma_0)r^2u + \lambda u + (G_{13} + E_{13})rw' = 0,$$
$$-(G_{13} + E_{13})ru' + E_{33}^* w'' - (G_{13} + \sigma_0)r^2 w + \lambda w = 0, \quad r^2\sigma_0 = r_1^2\sigma_1^0 + r_2^2\sigma_2^0 \tag{2.7}$$

$$G_{13}^*\upsilon'' - (G_{12} + \sigma_0)r^2\upsilon + \lambda\upsilon = 0 \tag{2.8}$$

When $\sigma_3^0 \neq 0$ the conditions on the free surface have the form

$$\sigma_{i3}^* = \sigma_{i3} + \sigma_3^0 u_i' = 0 \text{ when } z = 0, \quad i = 1, 2, 3 \tag{2.9}$$

and after changing to the unknowns (2.6), conditions (2.9) take the form

$$\sigma^* = G_{13}^* u' + G_{13}rw = 0, \quad \sigma_{33}^* = -E_{13}ru + E_{33}^* w' = 0, \quad \tau^* = G_{13}^*\upsilon' = 0 \tag{2.10}$$

Eq. (2.8) does not have non-trivial solutions which satisfy boundary conditions (2.2) and (2.10).

The solution of system (2.7), satisfying condition (2.2), has the form

$$w(z) = \sum_{k=1,2} C_k e^{\alpha_k rz}, \quad u(z) = \sum_{k=1,2} C_k b_k e^{\alpha_k rz}, \quad b_k = \frac{E_{33}^*\alpha_k^2 - G_{13} + \lambda^*}{(E_{13} + G_{13})\alpha_k} \tag{2.11}$$

where

$$\lambda^* = \frac{\lambda}{r^2} - \sigma_0, \quad \mathrm{Re}(\alpha_k) > 0 \tag{2.12}$$

C_k are constants and α_k are the roots of the characteristic equation of system (2.7)

$$a_0\alpha^4 - a_1\alpha^2 + a_2 = 0 \tag{2.13}$$

where

$$a_0 = G_{13}^*E_{33}^*, \quad a_1 = E_{33}^*(E_{11} - \lambda^*) + G_{13}^*(G_{13} - \lambda^*) - (E_{13} + G_{13})^2,$$
$$a_2 = (E_{11} - \lambda^*)(G_{13} - \lambda^*) \tag{2.14}$$

Eq. (2.13) has two roots with $\mathrm{Re}(\alpha_k) > 0$ when $a_2 > 0$ and $a_0 > 0$.

If (in the two-dimensional case) we seek a solution of the initial system (1.1) in the form

$$u(x, z, t) = u(z)\cos(rx - \omega t), \quad w(x, z, t) = w(z)\sin(rx - \omega t) \tag{2.15}$$

then, when conditions (2.2) and (2.10) are satisfied, we obtain a Rayleigh wave propagating on the surface of the half-space with velocity $v = \sqrt{\lambda^*/\rho}$.

Satisfying conditions (2.10) on the free surface, we obtain a dispersion equation both for the frequency of the vibrations (2.1) localized close to the surface, and for the Rayleigh wave

$$(G_{13}^*\alpha_1 b_1 + G_{13})(E_{33}^*\alpha_2 - E_{13}b_2) - (G_{13}^*\alpha_2 b_2 + G_{13})(E_{33}^*\alpha_1 - E_{13}b_1) = 0 \tag{2.16}$$

In the case of an isotropic body, Eq. (2.16) also takes the form

$$(2G - \lambda^*)^2 - (2G + \sigma_3^0)^2\alpha_1\alpha_2 = 0 \tag{2.17}$$

The values of the dimensionless frequency parameter $\Lambda^h = \lambda^*/G$ for a number of values of ν and $\sigma_3^* = \sigma_3^0/G$ are shown in Table 1.

Consider a transversely isotropic foundation with low moduli of elasticity in the vertical direction

$$\{E_{33}, E_{13}, G_{13}\} \ll \{E_{11}, G_{12}\} \tag{2.18}$$

Eq. (2.16) then gives approximately

$$\lambda^* = G_{13}^*(1 - \Delta), \quad \Delta = O(E_{33}/E_{11}) \tag{2.19}$$

Table 1

ν	$\sigma_3^* = 0.5$	0	−0.5	−0.8	−0.9	−0.98
0.0	0.856	0.764	0.500	−0.151	−0.972	−4.721
0.1	0.878	0.798	0.566	−0.010	−0.692	−4.089
0.2	0.899	0.830	0.629	0.126	−0.512	−3.421
0.3	0.918	0.860	0.687	0.253	−0.292	−2.723
0.4	0.937	0.888	0.739	0.367	−0.088	−2.020
0.5	0.954	0.896	0.784	0.466	0.089	−1.334

Table 2

$e = \frac{E_1}{E_2}$	$\eta = 0.5$	0	−0.5	−0.8	−0.9	−0.98
1	0.918	0.860	0.687	0.245	−0.298	−2.723
10	0.978	0.963	0.912	0.758	0.512	−1.100
100	0.998	0.996	0.991	0.974	0.946	0.724

To illustrate this, consider a foundation consisting of alternating isotropic layers with parameters (1.5). After averaging, we obtain a transversely isotropic foundation with parameters given by formulae (1.6). We will take

$$\nu_1 = \nu_2 = 0.3, \quad E_1/E_1 = e, \quad h_1 = h_2$$

and for a series of values of e and $\eta = \sigma_3^0/G_{13}$ we will obtain the value of the frequency parameter $\Lambda^h = \lambda^*/G_{13}$. The results are shown in Table 2. Our aim is to follow how the results depend on the measure of the anisotropy of the material. Here and henceforth we will take the values of the parameters (1.5) indicated above, for which the measure of the anisotropy is the quantity e, and the values of the remaining parameters are fixed. Strong anisotropy can, of course, also be obtained for other combinations of the parameters (1.5), but their consideration would unjustifiably increase the length of this paper.

We can see that, as the parameter e increases, the frequency parameter Λ approaches unity, which serves as a confirmation of the approximate asymptotic formula (2.19).

Rayleigh waves, propagating on the surface of an elastic isotropic half-space have been investigated in detail.[3] Eq. (2.17) enables us to investigate the effect of the initial stresses on the velocity v of a Rayleigh wave for an isotropic half-space, while Eq. (2.16) enables us to investigate this for a transversely isotropic half-space. If the direction of the orthotropy coincides with the direction of propagation of the Rayleigh wave, Eq. (2.16) can also be used for an orthotropic half-space. Here, if the initial stresses $\sigma_1^0 = \sigma_2^0 = \sigma_0$, the frequency $\omega = r\sqrt{(\Lambda^h G_{13} + \sigma_0)/\rho}$ only depends on r. If $\sigma_1^0 \neq \sigma_2^0$, the frequency $\omega = \sqrt{(r^2 \Lambda^h G_{13} + \sigma_1^0 r_1^2 + \sigma_2^0 r_2^2)/\rho}$ depends on the wave numbers r_1 and r_2 separately. The frequency parameter Λ^h depends only on the properties of the material and on the initial stress and is independent σ_3^0 of the initial stress σ_0, which occurs explicitly in the formula for ω.

The conditions for the material to be stable are the following inequalities

$$-\sigma_0 < G_{13}, \quad -\sigma_3^0 < \min\{G_{13}, E_{33}\} \tag{2.20}$$

When conditions (2.20) are satisfied, surface vibrations and Rayleigh waves are only possible when $\lambda > 0$, i.e., when

$$-\sigma_0 < \Lambda^h G_{13} \tag{2.21}$$

As follows from Tables 1 and 2, for certain values $\sigma_3^* < 0$ and other parameters of the problem, we have $\Lambda^h < 0$. In this case surface vibrations and Rayleigh waves are possible by virtue of condition (2.21) only for fairly large initial tensile stresses σ_0.

For certain values of the initial stresses σ_0 and σ_3^0 the half-space loses stability and waves are formed on the free surface. Suppose $\Lambda_*^h < 1$ is the root of Eq. (2.16). Then the boundary of the stability region will be $G_{13}\Lambda_*^h = -\sigma_0$. The effect of a reduction in the critical load with a localization of the form of the bending close to the free edge or the free surface is well known. It has been described for the compression of a plate with a free edge,[4] for shells[5] and for an isotropic half-space.[6]

3. The reaction of the elastic foundation

We will consider below the vibrations of a plate lying on the transversely isotropic foundation considered in Section 2. Here it is necessary to know the reaction of the foundation to a displacement of its surface. A solution of this problem was obtained earlier[7,8] in certain special cases.

Suppose we are given the displacements $u(0) = u_0$ and $w(0) = w_0$ on the surface $z = 0$. Then, in formulae (2.11)

$$C_1 = \frac{u_0 - b_2 w_0}{b_1 - b_2}, \quad C_2 = \frac{b_1 w_0 - u_0}{b_1 - b_2} \tag{3.1}$$

From formulae (2.10) we obtain the stresses $\sigma^*(0)$ and $\sigma'_{33}(0)$, generated by these displacements

$$\sigma^*(0) = r(c_{11}u_0 + c_{13}w_0), \quad \sigma_{33}^*(0) = r(c_{13}u_0 + c_{33}w_0) \tag{3.2}$$

where, for a transversely isotropic material, the stiffnesses of the foundation c_{ij} have the form

$$c_{11} = \frac{G^*(\alpha_1 b_1 - \alpha_2 b_2)}{b_1 - b_2}, \quad c_{13} = \frac{G_{13}^* b_1 b_2(\alpha_2 - \alpha_1)}{b_1 - b_2} + G_{13}, \quad c_{33} = \frac{E_{33}^*(b_1\alpha_2 - b_2\alpha_1)}{b_1 - b_2} \tag{3.3}$$

The difference between the coefficients (3.3) and those obtained previously in Refs 7 and 8 is the fact that, in particular, the forces of inertia of the foundation and the initial stresses σ_3^0 are taken into account.

The displacement of the surface $v(0) = v_0$, by virtue of relations (2.8) and (2.10), generate a stress

$$\tau^*(0) = G_{13}^*\alpha_3 r v_0, \quad \alpha_3 = \sqrt{(G_{12} - \lambda^*)/G_{13}^*} \tag{3.4}$$

where α_3 is the root of the characteristic equation for Eq. (2.8).

4. Formulation of the problem of the vibrations of a plate lying on elastic foundation

The Kirchhoff–Love approach. We will consider the free vibrations of a plate of thickness h, made of an elastic transversely isotropic material and lying on the transversely isotropic foundation considered above. There is a rigid contact between the plate and the foundation. The plate and the foundation are under conditions of vertical compression with a constant stress σ_3^0 and a horizontal omnidirectional tension (compression) with a constant deformation ε_0 (for a compression $\sigma_3^0 < 0$ and $\varepsilon_0 < 0$). Here the initial stresses σ_0 for the foundation and the plate are, respectively,

$$\sigma_0^f = \frac{E_0^f}{1-\nu^f}\varepsilon_0 + \frac{E_{13}^f}{E_{33}^f}\sigma_3^0, \quad \sigma_0^p = \frac{E_0^p}{1-\nu^p}\varepsilon_0 + \frac{E_{13}^p}{E_{33}^p}\sigma_3^0; \quad E_0^{(f,p)} = \frac{E^{(f,p)}}{1-(\nu^{f,p})^2} \tag{4.1}$$

Here and below the superscript f is used for quantities describing the foundation, and p is used for the like quantities of the plate.

The equation of the vibrations of the plate, according to the Kirchhoff–Love model, has the form

$$D^p\Delta\Delta w - T^p\Delta w + \sigma_{33}^* + \rho^p\frac{\partial^2 w}{\partial t^2} = 0$$

$$D^p = \frac{E_0^p h^3}{12}, \quad T^p = h\sigma_0^p, \quad E_0^p = \frac{E^p}{1-(\nu^p)^2}, \quad \Delta w = \frac{\partial^2 w}{\partial x_1^2} + \frac{\partial^2 w}{\partial x_2^2} \tag{4.2}$$

Here E^p, ν^p and ρ^p are Young's modulus, Poisson's ratio and the density of the material of the plate, T^p is the initial force in the plate and σ_{33}^* is the pressure of the foundation on the plate.

Suppose the plate performs flexural vibrations

$$w(x_1, x_2, t) = w_0\sin(r_1x_1)\sin(r_2x_2)e^{i\omega t}$$

where r_1 and r_2 are specified wave numbers, $\sigma_{33}^* = rc_{33}w_0$ and ω is the required frequency.

The dispersion equation has the form

$$\frac{\mu^4}{12} + \mu^2\varepsilon + \mu\hat{c}_{33} - \Lambda^k = 0 \tag{4.3}$$

where

$$\mu = rh = \frac{2\pi h}{L}, \quad \hat{c}_{33} = \frac{c_{33}}{E_0^p} \simeq \frac{G_{13}^f}{E_0^p}, \quad \Lambda^k = \frac{\rho^p h^2\omega^2}{E_0^p}, \quad \varepsilon = \varepsilon_0(1+\nu^p) + \frac{E_{13}^p\sigma_3^0}{E_{33}^p E_0^p} \tag{4.4}$$

Here L is the wavelength on the plate surface. For natural assumptions, Eq. (4.3) contains three small parameters μ, $\hat{c}_{33}$ and ε. The frequency of the vibrations ω occurs explicitly in this equation via the frequency parameter Λ^k and implicitly via the coefficients c_{33}, which take into account the inertia forces of the foundation. The use of Eq. (4.3) is limited by two facts. First, if the effective initial compressive strain ε is large, the plate loses stability. The criterion of stability is the inequality $\Lambda^* > 0$, which gives

$$-\varepsilon < \min_{\mu}\left(\frac{\mu^2}{12} + \frac{\hat{c}_{33}}{\mu}\right) = \frac{(6\hat{c}_{33})^{2/3}}{4}, \quad \mu = (6\hat{c}_{33})^{1/3} \tag{4.5}$$

The second limitation, imposed on the frequency parameter Λ^k, arises from the requirement that the characteristic Eq. (2.13) for the foundation should not have pure imaginary roots, since otherwise the energy of the vibrations will become infinite and free non-decaying vibrations of the plate (and of the Rayleigh wave also) will be impossible. This requirement leads to the inequality

$$\Lambda^k < \mu^2\frac{\rho^p}{\rho^f}\frac{G_{13}^f + \sigma_3^0}{E_0^p} \tag{4.6}$$

where ρ^f is the density of the foundation material and G_{13}^f is its stiffness to transverse shear.

5. The Timoshenko–Reissner approach

In the case of considerable anisotropy, the two-dimensional Timoshenko–Reissner model gives much more accurate results than the Kirchhoff–Love model. In the Timoshenko–Reissner model for transverse vibrations the normal element of the plate has three degrees of freedom: a vertical displacement w and two angles of rotation φ_1 and φ_2.

The equations of motion of this element have the form

$$\frac{\partial Q_1}{\partial x_1}+\frac{\partial Q_2}{\partial x_2}+T^p\Delta w-\sigma_{33}^*-\rho^p h\frac{\partial^2 w}{\partial t^2}=0$$

$$D^p\left(\frac{\partial^2\varphi_k}{\partial x_k^2}+\frac{1-\nu^p}{2}\frac{\partial^2\varphi_k}{\partial x_{3-k}^2}+\frac{1+\nu^p}{2}\frac{\partial^2\varphi_2}{\partial x_1\partial x_2}\right)-Q_k-\frac{h\sigma_{k3}^*}{2}-\frac{\rho^p h^3}{12}\frac{\partial^2\varphi_k}{\partial t^2}=0,\quad k=1,2 \tag{5.1}$$

where we have used the same notation as in Eq. (4.2), and Q_1 and Q_2 are shearing forces, related to the shear angles δ_1 and δ_2 by the formulae

$$Q_k=\Gamma h\delta_k,\quad \delta_k=\varphi_k+\frac{\partial w}{\partial x_k},\quad k=1,2,\quad \Gamma=\gamma G_{13}^p \tag{5.2}$$

The factor γ takes into account integrally the nonuniformity of the distribution of the shear stresses over the shell thickness and is taken below to be equal to $\gamma=5/6$. The value $\gamma=5/6$ is asymptotically accurate when the foundation is fairly soft compared with the plate, while the waves are long compared with its thickness.[9,10] In system (5.1) the stresses at the interface of the plate and the foundation are denoted by σ_{i3}^*.

After introducing the new unknown functions Θ and Ψ instead of the angles φ_1 and φ_2

$$\varphi_1=\frac{\partial\Theta}{\partial x_2}-\frac{\partial\Psi}{\partial x_1},\quad \varphi_2=-\frac{\partial\Theta}{\partial x_1}-\frac{\partial\Psi}{\partial x_2} \tag{5.3}$$

system (5.1) can be split (as in Section 2) into an equation in the function Θ

$$\frac{1-\nu^p}{2}D^p\Delta\Theta-\Gamma h\Theta-\frac{h\tau^*(0)}{2}-\frac{\rho^p h^3}{12}\frac{\partial^2\Theta}{\partial t^2}=0 \tag{5.4}$$

and a system of equations in the functions w and Ψ

$$\Gamma h(\Delta w-\Delta\Psi)+T\Delta w-\sigma_{33}^*(0)-\rho^p h\frac{\partial^2 w}{\partial t^2}=0$$

$$-D^p\Delta\Delta\Psi-\Gamma h(\Delta w-\Delta\Psi)-\frac{hr\sigma^*(0)}{2}+\frac{\rho^p h^3}{12}\frac{\partial^2\Delta\Psi}{\partial t^2}=0 \tag{5.5}$$

Here $\tau^*(0)$, $\sigma_{33}^*(0)$ and $\sigma^*(0)$ are the stresses (3.4) and (3.3) on the plate-foundation interface, which appears as a result of vertical displacements w and horizontal displacements, related to the angles of rotation of the normal φ_1 and φ_2 around the horizontal axes (in view of the smallness of the thickness h the latter stresses, as a rule, are ignored).

Assuming in Eq. (5.4) that

$$\Theta=\Theta_0\cos r_1x_1\cos r_2x_2e^{i\omega t}$$

and taking formula (3.4) into account, we obtain the following dispersion equation for the frequency ω

$$\frac{\rho^p h^3\omega^2}{12}=\Gamma h+\frac{D^p(1-\nu^p)r^2}{2}+\frac{(G_{13}^f+\sigma_3^0)h^2}{4}\sqrt{\frac{(G_{12}^f+\sigma_0^f)r^2-\rho^f\omega^2}{G_{13}^f+\sigma_3^0}} \tag{5.6}$$

As a rule, Eq. (5.6) has no real roots. In fact, the third term on the right-hand side is small. Dropping this term, we obtain the frequency ω, and substituting this we obtain a negative value for the radicand f

$$\left(\frac{10G_{13}^p}{\mu^2}+\frac{E^p(1-\nu^p)}{2}\right)\frac{\rho^f}{\rho^p}>G_{12}^f+\sigma_0^f \tag{5.7}$$

If inequality (5.7) is satisfied, free non-decaying vibrations of this type are impossible, since the energy of the vibrations is radiated in the direction $z\to-\infty$.

In system (5.5), describing the transverse vibrations, we put

$$w=w_0\sin r_1x_1\sin r_2x_2e^{i\omega t},\quad \Psi=\Psi_0\sin r_1x_1\sin r_2x_2e^{i\omega t}$$

Then, in formulae (3.2) $u_0 = -\mu\Psi_0/2$ and the dispersion can be written in the form

$$(\Lambda^t - g\mu^2 - \varepsilon\mu^2 - \hat{c}_{33}\mu)\left(\frac{\Lambda^t - \mu^2}{12} - g - \frac{\hat{c}_{11}\mu}{4}\right) - \left(g - \frac{\hat{c}_{13}\mu}{2}\right)\left(g\mu^2 - \frac{\hat{c}_{13}\mu}{2}\right) = 0 \tag{5.8}$$

where we have used the same notation as in Eq. (4.3), and also

$$g = \frac{\Gamma}{E_0^p}, \quad \hat{c}_{ij} = \frac{c_{ij}}{E_0^p}, \quad \Lambda^t = \frac{\rho^p h^2 \omega^2}{E_0^p} \tag{5.9}$$

As for a Timoshenko beam (see, for example, Ref.11), Eq. (5.8) has two roots. If the wavelength L is large compared with the plate thickness h and $\Gamma/E^p \ll 1$, this equation containssmall parameters μ, ε, g, c_{ij}^*. Assuming that they are related by the serial relations

$$\varepsilon \sim \mu^2, \quad \hat{c}_{ij} \sim \mu^3, \quad g \sim \mu^2 \tag{5.10}$$

we obtain approximately for the smaller and larger roots of Eq. (5.8)

$$\Lambda_1^t = \frac{\mu^4 g}{12g + \mu^2} + \mu^2\varepsilon + \mu\hat{c}_{33}, \quad \Lambda_2^t = \mu^2 + 12g \tag{5.11}$$

If the stiffness of the plate for transverse shear is not small ($12g \gg \mu^2$), formula (5.11) for Λ_1^t changes into equality (4.3), while the stability condition

$$-\varepsilon < \min_{\mu} f(\mu, g, \hat{c}_{33}) = f_0(g, \hat{c}_{33}), \quad f(\mu, g, \hat{c}_{33}) = \frac{\mu^2 g}{12g + \mu^2} + \frac{\hat{c}_{33}}{\mu} \tag{5.12}$$

becomes condition (4.5), obtained for the Kirchhoff–Love model.

If

$$g > g_* = (\hat{c}_{33}^2/3)^{1/3}, \quad \text{i.e.} \quad \hat{c}_{33} < \sqrt{3g^3} \tag{5.13}$$

the function $f(\mu, g, \hat{c}_{33})$ has a minimum $f_0 < g$ for a certain finite value of μ. When the stiffness of the base $\hat{c}_{33}$ increases to $\hat{c}_{33} = \sqrt{3g^3}$ the value of f_0 increases to $f_0 = g$, and μ increases to the value $\mu = \sqrt{12g}$. If inequality (5.13) breaks down, the value of μ, for which a minimum $f_0 = g$ is reached, jumps to $\mu = \infty$. According to the two-dimensional Timoshenko–Reissner model, breakdown of condition (5.13) indicates a loss of stability of the plate material. Condition (5.13) will be made more accurate for the three-dimensional model in Section 7.

6. Longitudinal vibrations of the plate

When integrating system of equations (2.7) numerically between the frequencies Λ_1^p and Λ_2^p for fixed wave numbers r_1 and r_2 we have the frequency of longitudinal vibrations of the plate available. The system of equations of longitudinal vibrations of a transversely isotropic plate on an elastic foundation, within the framework of the hypothesis of a plane stress state, has the form

$$\begin{aligned} &(E_{11}^p + \sigma_0^p)\frac{\partial^2 u_1}{\partial x_1^2} + (G_{12}^p + \sigma_0^p)\frac{\partial^2 u_1}{\partial x_2^2} + (E_{12}^p + G_{12}^p)\frac{\partial^2 u_2}{\partial x_1 \partial x_2} + \frac{c_{11}}{h}\frac{\partial u_1}{\partial x_1} - \rho^p\frac{\partial^2 u_1}{\partial t^2} = 0 \\ &(E_{12}^p + G_{12}^p)\frac{\partial^2 u_1}{\partial x_1 \partial x_2} + (E_{11}^p + \sigma_0^p)\frac{\partial^2 u_2}{\partial x_2^2} + (G_{12}^p + \sigma_0^p)\frac{\partial^2 u_2}{\partial x_1^2} + \frac{c_{11}}{h}\frac{\partial u_2}{\partial x_2} - \rho^p\frac{\partial^2 u_2}{\partial t^2} = 0 \end{aligned} \tag{6.1}$$

We will seek a solution of this system in the form (2.1) with $w = 0$ and we will change to the unknown functions u and v using formulae (2.6). We obtain the following dispersion equation for the function u, which describes the tension-compression strains in the plane of the plate,

$$\Lambda^u = \frac{\rho^p h^2 \omega^2}{E_0^p} = \mu^2\left(1 + \frac{\sigma_0^p}{E_0^p}\right) + \mu\hat{c}_{11}, \quad \mu = rh \tag{6.2}$$

where the second term takes into account the stiffness of the foundation.

For the function v in formula (2.6), which describes longitudinal shear-torsional vibrations, the dispersion equation has the form

$$\Lambda^v = \frac{\rho^p h^2 \omega^2}{E_0^p} = \mu^2\left(\frac{1 - \nu^p}{2} + \frac{\sigma_0^p}{E_0^p}\right) + \mu\hat{c}_{22} \tag{6.3}$$

where $\hat{c}_{22}$ is a coefficient which takes into account the reaction of the foundation and is determined from relations (3.4). Formulae (6.2) and (6.3), as well as formulae (4.11), describe free non-decaying vibrations only when conditions (4.5) and (4.6) are satisfied.

Generally speaking, the transverse and longitudinal vibrations are coupled to one another through the foundation, by virtue of formulae of the type (3.2), but for a relatively weak foundation (when $E_{11}^f/E_{11}^p \ll 1$) this coupling can be neglected, as was done above.

7. Thickness vibrations of the plate

We will consider vibrations of the plate with the formation of several thickness waves. In this case, as a rule, inequality (4.6) breaks down. Hence, here we will assume that the foundation is non-inertial ($\rho^f = 0$).

In the three-dimensional formulation the vibrations of a transversely isotropic plate, after separation of the variables (2.1) and replacement (2.6), are described by the same Eqs. (2.7) and (2.8) with the corresponding values of the moduli of elasticity and density for the plate. After extending the scale $z = h\zeta$ we can write these equations in the dimensionless form

$$\hat{G}_{13}^* u'' - (\hat{E}_{11} + \hat{\sigma}_0)\mu^2 u + \Lambda u + (\hat{G}_{13} + \hat{E}_{13})\mu w' = 0$$

$$-(\hat{G}_{13} + \hat{E}_{13})\mu u' + \hat{E}_{33}^* w'' - (\hat{G}_{13} + \hat{\sigma}_0)\mu^2 w + \Lambda w = 0 \tag{7.1}$$

$$\hat{G}_{13}^* \upsilon'' - (\hat{G}_{12} + \hat{\sigma}_0)\mu^2 \upsilon + \Lambda \upsilon = 0 \tag{7.2}$$

The prime denotes a derivative with respect to ζ while the hat denotes that the corresponding quantity relates to E_0^p (i.e., $\hat{Z} = Z/E_0^p$) and the frequency parameter Λ is the same as in (4.4).

For small values of the parameter $\mu = rh$, system (7.1) can be split into separate equations for the unknown functions u and w. As a result, in the zero approximation in μ we obtain separate boundary-value problems for the functions u, υ and w

$$\hat{G}_{13}^* u'' + \Lambda_1 u = 0, \quad \hat{G}_{13}^* u'(0) = \mu \hat{c}_{11} u(0), \quad u'(1) = 0$$

$$\hat{G}_{13}^* \upsilon'' + \Lambda_2 \upsilon = 0, \quad \hat{G}_{13}^* \upsilon'(0) = \mu \hat{c}_{22} \upsilon(0), \quad \upsilon'(1) = 0$$

$$\hat{E}_{33}^* w'' + \Lambda_3 w = 0, \quad \hat{E}_{13}^* w'(0) = \mu \hat{c}_{33} w(0), \quad w'(1) = 0 \tag{7.3}$$

The stiffnesses of the base are $\hat{c}_{ii}$ calculated for $\omega = 0$. Problems (7.3) define three series of frequencies of high-frequency thickness vibrations with asymptotic

$$\Lambda_{j,n} = A_j\left(n^2\pi^2 + \frac{2\mu\hat{c}_{jj}}{A_j} + O(n^{-1})\right), \quad j = 1, 2, 3, \quad n = 1, 2, \ldots, \tag{7.4}$$

When $n = 0$ we are not justified in dropping terms with the small factor μ in Eqs. (7.1) and (7.2), while the approximate values of the parameter Λ^p can be found from the two-dimensional models considered in Sections 4–6. The parameter Λ^u (see (6.2)) corresponds to Λ_{10}, the parameter Λ^v (see (6.3)) corresponds to Λ_{20}, while the parameter Λ^k (see (4.4)) or the parameter Λ_1^t from (5.11) corresponds to Λ_{30}. Moreover, the parameter Λ_2^t from (5.11) is close to the value of Λ_{21}, with certain assumptions,[12] which is confirmed by the numerical results presented below. Thickness vibrations are not considered in more detail here.

8. Some numerical results

When discussing the numerical results we will compare the exact solution of the equations of the theory of elasticity with the approximate solutions obtained using the Kirchhoff–Love and Timoshenko–Reissner models. We will confine ourselves to considering system (2.7) or (7.1). Eqs. (2.8), (5.4) and (7.2) will not be considered. It is possible to construct an exact analytical solution, but it turned out to be more convenient to use a mixed method. We will integrate system (7.1) for the plate numerically in the layers $0 \le z \le 1$, and we will take the initial conditions in the Cauchy problem for $z = 0$ from the analytical solution (2.11) for the foundation. The arbitrary constants C_1 and C_2 and the frequency parameter Λ^p will be found by satisfying the boundary conditions when $z = 1$.

We will consider a transversely isotropic foundation and plate with parameters defined from formulae (1.6) for values of the parameters (1.5) which satisfy the relations

$$E_1^p = 10E_2^p = 100E_1^f = 1000E_2^f, \quad \nu^p = \nu^f = 0.3, \quad h_1^p = h_2^p, \quad h_1^f = h_2^f \tag{8.1}$$

where, as before, the superscripts p and f relate to the plate and the foundation. In view of relations (8.1) both the plate and the foundation are approximately 5 times stiffer in the horizontal direction than in the vertical direction. The foundation is 100 times softer than the plate. The remaining parameters of the problem will vary. Below we will discuss the effect of the strain wavelength, the horizontal initial strain, the vertical initial stress and the inertia of the foundation.

The effect of the strain wavelength L is described by the parameter $\mu = rh = 2\pi h/L$. We will assume here that there are no initial stresses, and the foundation is lag-free, i.e., $\varepsilon_0 = \sigma_3^0 = \rho^f = 0$. The values of a number of the parameters, which enable the effect of the strain wavelength to be estimated, are presented below

μ	0.03	0.1	0.3	1.0	3.0
Λ_1/μ^4	74.60	2.160	0.160	0.0467	0.0100
Λ^k/μ^4	74.46	2.092	0.158	0.0853	0.0834
Λ_1^t/μ^4	74.86	2.126	0.155	0.0462	0.0095
Λ_2/μ^2	1.109	1.032	1.009	0.982	1.007
Λ_1^u/μ^2	1.109	1.033	1.011	1.003	1.001
Λ_3	1.143	1.153	1.239	2.166	9.06
Λ_2^t	1.158	1.169	1.258	2.209	10.29

Table 3

η	$\mu = 0.03$	0.1	0.3	1.0
−0.99	41.9	1.20	0.115	0.0450
0.0	74.6	2.16	0.160	0.0467
1.0	105.7	3.10	0.205	0.0484

The quantities Λ_1, Λ^k and Λ_1^t are the parameters of the least frequency of the mainly flexural vibrations obtained in the numerical integration using the Kirchhoff–Love and the Timoshenko–Reissner models respectively. As the wavelength *L* is reduced compared with the plate thickness *h*, the region of applicability of the Kirchhoff–Love model becomes exhausted, and the Timoshenko–Reissner model gives satisfactory results over the whole range of values of μ considered. The quantities Λ_2 and Λ_1^u give higher-frequency vibrations of the plate with displacements in its plane. They were obtained by numerical integration and using formula (6.2) respectively, and agree well with one another. The value of the third frequency of the mainly shear vibrations is described by the parameters Λ_3 and Λ_2^t, obtained numerically and from formula (5.11) respectively, which gives the second frequency of the vibrations using the Timoshenko–Reissner model. The error of formula (5.11) increases as μ increases.

The effect of a horizontal initial strain ε_0. The plate loses stability at a fairly high level of initial compressing strain. As above, we will assume that $\sigma_3^0 = \rho^f = 0$. Integration of system (7.1) gives the following critical values of the compressive strain and the wave-forming parameter: $\varepsilon_0 = -0.01015$ and $\mu = 0.23$. For comparison we mention that the Kirchhoff–Love model gives $\varepsilon_0 = -0.00947$ and $\mu = 0.22$, while the Timoshenko–Reissner model gives $\varepsilon_0 = -0.00934$ and $\mu = 0.23$. For the assumed values of the parameters (8.1), condition (5.13) is satisfied with a large margin. The first three eigenvalues of the parameter Λ, obtained from (7.1) for $\mu = 0.23$, are given below

ε_0	−0.01	0.0	0.01
Λ_1/μ^4	0.004	0.259	0.514
Λ_2/μ^2	1.000	1.013	1.026
Λ_3	1.199	2.000	2.000

As might have been expected, when formulae (2.11) are taken into account, the dependence of the parameters Λ_i on the initial strain ε_0 is linear, the dependence of Λ_1 on ε_0 is considerable, the value of Λ_2 is weak, while the quantity Λ_3 is practically independent of ε_0.

We will consider one other problem on the loss of stability of a plate in the case when the right- and left-hand sides of inequality (5.12) are close to one another. Instead of relations (8.1) we take

$$E_1^f = 0.07E_1^p, \quad E_2^f = 0.06E_1^p, \quad \nu^p = \nu^f = 0.3, \quad h_1^p = h_2^p, \quad h_1^f = h_2^f \tag{8.2}$$

and we also let $\sigma_3^0 = 0$. The ratio $e = E_2^p/E_1^p$ will change in order to approximate to the critical situation. We show below, for a number of values of *e*, the results of calculations using the three-dimensional model, we give the critical strain ε corresponding to the value of the wave parameter μ, and the values of *g* and g_* calculated from formulae (5.9) and (5.13)

e	0.094	0.095	0.1	0.12	0.15
$-\varepsilon$	0.0843	0.0812	0.0768	0.749	0.743
μ	9.00	3.95	1.89	1.30	1.08
g	0.092	0.092	0.096	0.112	0.132
g_*	0.111	0.111	0.111	0.110	0.110

According to the Timoshenko–Reissner model, the loss of stability for a finite value of μ occurs when $g > g_*$. It can be seen that this condition is satisfied when $e \geq 0.12$. For smaller values of *e* the inequality $g > g_*$ breaks down, and there is a rapid increase in the parameter μ, the wavelength becomes considerably shorter than the plate thickness, and the form of the bending is localized close to the outer surface of the plate. When $e \simeq 0.094$ there is a loss of stability of the plate material. Unlike the Timoshenko–Reissner model, here the transition to instability does not occur suddenly, but increases in proportion to the parameter *e*.

The effect of a vertical initial stress σ_3^0. We will assume that relations (8.1) are satisfied, and we will also put $\sigma_0 = \rho^f = 0$, $\sigma_3^0 = G_{13}^f\eta$, where $\eta > -1$. As noted in Section 2, when $\eta \leq -1$ the material of the foundation loses stability. Table 3 shows values of the quantity $\mu^{-4}\Lambda_1(\eta, \mu)$ for a number of values of η and μ, obtained by numerical integration of system (7.1).

The dependence of Λ_1 on η is close to linear up to the critical value $\eta = -1$. Unlike the case of an initial horizontal strain ε_0, here the parameter Λ_1 varies over a narrow range as the load changes. The values of Λ_2 and Λ_3 change only slightly as η changes in the range considered.

The effect of the inertia forces of the foundation. We will use the data given in (8.1) and we will assume that there are no initial stresses, i.e., $\varepsilon_0 = \sigma_3^0 = 0$. We will vary the density ratio $\rho_0 = \rho^f/\rho^p$ and the wave number μ. By virtue of formula (4.6), the parameter ρ_0 will be bounded by the inequality

$$\rho_0 < \rho_* = \frac{\mu^2 \hat{G}_{13}^f}{\Lambda^*}, \quad \hat{G}_{13}^f = 0.001157 \tag{8.3}$$

violation of which means that the spectrum of the oscillations is not real. In Table 4 we show the first two eigenvalues, where $\Lambda_k^0 (k = 1, 2)$ are obtained for $\rho_0 = 0$, while Λ_k^* is obtained for the maximum possible value of $\rho_0 = \rho_*$, for the smaller value of which Λ_k is real.

The effect of inertia forces of the foundation can be estimated by comparing the quantities Λ_k^0 and Λ_k^*. For a thinner plate (for smaller μ) this effect is greater, and for Λ_1 it is greater than for Λ_2. The results for Λ_3 are not considered, since in this case $\rho_* \sim 10^{-5}$.

Table 4

μ	Λ_1^0	Λ_1^*	ρ_*	Λ_2^0	Λ_2^*	ρ_*
0.03	$6.04\cdot10^{-5}$	$3.38\cdot10^{-5}$	0.021	$9.98\cdot10^{-4}$	$9.90\cdot10^{-4}$	0.00104
0.1	$2.16\cdot10^{-4}$	$1.21\cdot10^{-4}$	0.069	$1.03\cdot10^{-2}$	$1.03\cdot10^{-2}$	0.00112
0.3	$1.30\cdot10^{-3}$	$1.00\cdot10^{-3}$	0.106	$9.08\cdot10^{-2}$	$9.07\cdot10^{-2}$	0.00114
1.0	$4.67\cdot10^{-2}$	$4.51\cdot10^{-2}$	0.255	$9.82\cdot10^{-1}$	$9.82\cdot10^{-1}$	0.00115

9. Discussion

This paper was the subject of a discussion at a conference (Ref. 13), devoted to surface and edge waves. In view of the theme of the conference the frequency range considered was limited to frequencies for which there was no radiation into the half-space, and the frequency spectrum is real. At higher frequencies, radiation occurs and free non-decaying vibrations are impossible, but when investigating forced vibrations at such frequencies at resonances, the amplitudes of the vibrations are finite. The dispersion equation for the vibrations is identical with the equation describing a Rayleigh wave.

A solution is sought in the form of a doubly periodic function of the surface coordinates, as a result of which the problem reduces to a one-dimensional problem while for a half-space an explicit analytical solution is constructed, which, in the case of a plate, is used to calculate the coefficients of the bed. For a plate of finite dimensions, the solution constructed can be considered as an inner solution, which must be supplemented by integrals of the edge effect, as was done in Ref.14 for shallow shells, rectangular in plan. However, due to the presence of a foundation in the problem considered here, an analytical construction of the edge effect is hardly possible.

Due to the transversal isotropy we can separate from the overall sixth-order problem a second-order problem describing the shear-torsion displacement in the horizontal plane. This problem, in which the vibrations, as a rule, are accompanied by radiation, is not considered here. The remaining fourth-order problem is similar to the problem of vibrations with displacements lying in one vertical plane.

This problem is considered when there are initial horizontal stresses in the foundation (σ_0^f) and in the plate (σ_0^p) and initial vertical stresses (σ_3^0), the same for the foundation and for the plate. When considering the vibrations it is necessary to confine ourselves to initial stresses (compressions) that are less in modulus than the critical values for which a loss of stability occurs. Two forms of instability are possible—instability of the material and surface instability. For the materials considered here, the moduli of elasticity of which satisfy the inequalities

$$E_{11}^p > E_{33}^p > G_{33}^p > E_{11}^f > E_{33}^f > G_{33}^f$$

the conditions for the stability of the material of the foundation and of the plate are $\min(\sigma_0^f, \sigma_3^0) + G_{33}^f > 0$ and $\min(\sigma_0^p, \sigma_3^0) + G_{33}^p > 0$ respectively. When at least one of these conditions breaks down the material loses stability, since the Hadamard condition[15] is not satisfied.

When there is no plate, a surface loss of stability of a transversely isotropic half-space is possible for initial stresses that are less in modulus than the limit values G_{33}^f. The wavelength on the surface when there is a loss of stability is impossible to determine within the framework of the formulation considered (a more detailed investigation of surface phenomena is necessary). If the initial stresses lie in the stability region, there is one frequency of surface vibrations and a Rayleigh wave corresponding to it. The frequency is inversely proportional to the wavelength on the surface, while the square of the frequency depends linearly on the stress σ_0^f.

When a plate is present, surface stability is the stability of a compressed transversely isotropic plate on an elastic foundation, complicated by the presence of initial vertical stresses. The wavelength on the plate surface is then defined uniquely. In the case of waves, the wavelength of which is long compared with the thickness ($\mu \ll 1$), the two-dimensional Kirchhoff–Love and Timoshenko–Reissner theories are applicable. An extremal situation is also possible here when this length becomes commensurable with the plate thickness or less than it ($\mu \sim 1$). In this case localization of the form of the bending close to the free surface of the plate is observed, and two-dimensional theories are inapplicable.

For long waves ($\mu \ll 1$) the lowest vibration frequency is well described by the Kirchhoff–Love and Timoshenko–Reisssner models. As μ increases the error of the Kirchhoff–Love model increases rapidly while the range of applicability of the Timoshenko–Reissner model is considerably wider, particularly for plates with strong anisotropy. The second frequency is close to the frequency of vibrations of a plate with displacements in its plane. The third frequency in value is close to the second frequency, which the Timoshenko–Reissner model gives.

Acknowledgement

This research was financed by the Russian Foundation for Basic Research (07-01-00250a).

References

1. Radionova VA, Titayev BF, Chernykh KR. *The Applied Theory of Anisotropic Plates and Shells*. St Petersburg: Izd SP Univ; 1996.
2. Bolotin VV, Novichkov YuN. *The Mechanics of Laminated Structures*. Moscow: Mashinostroyeniye; 1980.
3. Love AEH. *A Treatise on the Mathematical Theory of Elasticity*. Cambridge: Univ. Press; 1927.
4. Ishlinskii AYu. A limit transition in the theory of the stability of rectangular elastic plates. *Dokl Akad Nauk SSSR* 1954;**95**(3):477–9.
5. Tovstik PYe. *The Stability of Thin Shells*. Moscow: Nauka. Fizmatlit; 1995.
6. Morozov NF, Paukshto MV, Tovstik PYe. The stability of the surface layer under thermal loading. *Izv Ross Akad Nauk MTT* 1998;**1**:130–9.
7. Il'gamov MA, Ivanov VA, Gulin BV. *Strength, Stability and Dynamics of Shells with an Elastic Filler*. Moscow: Nauka; 1977.
8. Tovstik PYe. The reaction of a prestressed orthotropic foundation. *Vestnik St Petersburg Univ Ser 1* 2006;**4**:98–108.
9. Tovstik PYe. The local stability of plates and shallow shells on an elastic foundation. *Izv Ross Akad Nauk MTT* 2005;**1**:147–60.
10. Tovstik PE, Tovstik TP. On the 2D models of plates and shells including the transversal shear. *ZAMM* 2007;**87**:160–71.
11. Birger IA, Panovko YaG, editors. *Strength, Stability and Vibrations. Vol. 3*. Moscow: Mashinostroyeniye; 1968.

12. Tovstik PYe. The asymptotic character of approximate models of beams, plates and shells. *Vestnik SPU Ser 1* 2007:49–54.
13. Tovstik PE. On the vibrations of pre-stressed anisotropic plates and shells lying on a pre-stressed anisotropic elastic foundation. EUROMECH Colloq. 481: Recent Advances in the Theory and Application of Surface and Edge Waves. Keele Univ. 2007, p. 38.
14. Bolotin VV. The edge effect in the vibrations of elastic shells. *Prikl Mat Mekh* 1960;**24**(5):831–42.
15. Ciarlet PG. *Mathematical Elasticity*. Amsterdam etc: North-Holland; 1988.

Translated by R.C.G.